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Sasaki-Einstein Twist of Kerr-AdS Black Holes

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Abstract

We consider Kerr-AdS black holes with equal angular momenta in arbitrary odd spacetime dimensions ≥ 5. Twisting the Killing vector fields of the black holes, we reproduce the compact Sasaki-Einstein manifolds constructed by Gauntlett, Martelli, Sparks and Waldram. We also discuss an implication of the twist in string theory and M-theory.

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Kerr-AdS black holes are characterized by mass, angular momenta and cosmological constant. In spacetime dimension d, the number of angular momenta is equal to the rank of the rotation group SO$(d-1)$. The five-dimensional Kerr-AdS black holes with two angular momenta were constructed in [1], and recently the general form in arbitrary dimension was found by using the Kerr-Schild ansatz [2].

On the other hand, the Wick rotation of the black holes leads to Riemannian metrics. However, the metrics in general do not extend smoothly to compact manifolds. In [3][4][2], it was shown that this can be achieved by taking a certain limit (Page limit) which enhances the isometry of the metric. Indeed, the infinite series of Einstein metrics on compact manifolds were explicitly constructed [4][2], and analyzed in detail in [5].

Recently, infinite series of Sasaki-Einstein metrics on compact manifolds were presented in [6][7]. It is expected that these metrics can be related to some Kerr-AdS black holes by a certain limit. Our aim in this letter is to clarify the relation between them.

We begin with the $(2n+3)$-dimensional Kerr-AdS black hole with a negative cosmological constant $(2n+2)\lambda < 0$ $(n \geq 1)$ as follows [1][2]:

$$\hat{g} = -\frac{\hat{W}(r)}{\hat{b}(r)}dt^2 + \frac{dr^2}{\hat{W}(r)} + r^2\left(g_{\mathbb{C}P^n} + \hat{b}(r)\left(d\psi + A + \hat{f}(r)dt\right)^2\right), \tag{1}$$

where

$$\begin{aligned}
\hat{W}(r) &= 1 - \lambda r^2 - \frac{2M(\delta^2 + \lambda J^2)}{r^{2n}} + \frac{2MJ^2}{r^{2n+2}} = (1-\lambda r^2)\hat{b}(r) - \frac{2M\delta^2}{r^{2n}}, \\
\hat{b}(r) &= 1 + \frac{2MJ^2}{r^{2n+2}}, \\
\hat{f}(r) &= \frac{1}{J}\left(1 - \frac{\delta}{\hat{b}(r)}\right).
\end{aligned} \tag{2}$$

The metric $g_{\mathbb{C}P^n}$ is the Fubini-Study metric on $\mathbb{C}P^n$ with a normalization $Ric_{\mathbb{C}P^n} = (2n+2)g_{\mathbb{C}P^n}$, and the 1-form A is the U(1) connection associated with the Kähler form $dA/2$ on $g_{\mathbb{C}P^n}$. The black hole is parameterized by the mass M, the angular momentum J and a trivial parameter δ. The parameter δ is related to the parameter β introduced in [8] as $\delta = -\lambda J^2\beta + 1$. This metric is a special case that all angular momenta are set to be equal.

The metric (1) reduces to the AdS metric at $r \to \infty$ because the metric of the circle

bundle over $\mathbb{C}P^n$ tends to the standard metric of S^{2n+1}.[♮] A horizon appears for sufficiently small J. If we set $\delta^2 = -\lambda J^2$, the $\hat{W}(r)$ does not have positive roots so that the curvature singularity at $r = 0$ is not screened by the horizon, and so is naked. As will be seen below, in the Euclidean picture this solution is shown to be related to the Sasaki-Einstein metrics.

The Euclidean Einstein metric with a positive cosmological constant $(2n+2)\lambda > 0$ is extracted from the Kerr-AdS black hole (1) by the substitution $t \to i\tau$ and $J \to iJ$:

$$g = \frac{W(r)}{b(r)}d\tau^2 + \frac{dr^2}{W(r)} + r^2\Big(g_{\mathbb{C}P^n} + b(r)\,(d\psi + A + f(r)d\tau)^2\Big), \tag{3}$$

where

$$\begin{aligned} W(r) &= 1 - \lambda r^2 - \frac{2M(\delta^2 - \lambda J^2)}{r^{2n}} - \frac{2MJ^2}{r^{2n+2}} = (1-\lambda r^2)b(r) - \frac{2M\delta^2}{r^{2n}}, \\ b(r) &= 1 - \frac{2MJ^2}{r^{2n+2}}, \\ f(r) &= \frac{1}{J}\left(1 - \frac{\delta}{b(r)}\right). \end{aligned} \tag{4}$$

The metric has the isometry SU$(n+1)\times$U$(1)\times\mathbb{R}$. The generator of U$(1)\times\mathbb{R}$ is given by $(\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\tau})$. It is easy to see that under the Page limit and a special choice of the parameters [3][4][2] this metric reduces to a homogeneous Einstein metric with the isometry SU$(n+1)\times$SU$(2)\times$U(1) on a circle bundle over $\mathbb{C}P^n \times S^2$. Indeed, the metric can be written as

$$g_0 = \frac{1}{W_0}(d\chi^2 + \sin^2\chi d\eta^2) + r_0^2\left(g_{\mathbb{C}P^n} + b_0(d\psi + A + \frac{k}{2}\cos\chi d\eta)^2\right), \tag{5}$$

where

$$\begin{aligned} W_0 &= \frac{2\lambda(n+1)\,(2(n+1) - (n+2)b_0)}{n+1-b_0}, \\ r_0^2 &= \frac{n+1-b_0}{\lambda(n+1)}, \\ k &= \pm\frac{2\sqrt{(n+1)b_0(1-b_0)}}{b_0(2(n+1)-(n+2)b_0)} \in \mathbb{Z}\,, \end{aligned} \tag{6}$$

and b_0 is a constant with $0 < b_0 < 1$. In the case of $n = 1$, this reproduces the metric given in Theorem 2 of [4]. Further, for $k = 1$, it gives the homogeneous Sasaki-Einstein manifold $T^{1,1}$.

[♮] If we replace the Fubini-Study $\mathbb{C}P^n$ by an arbitrary Einstein-Kähler manifold with the same scalar curvature, we obtain another Kerr black hole with different asymptotic behavior.

We now transform the metric to inhomogeneous Sasaki-Einstein metrics on circle bundles over $\mathbb{C}P^n \tilde{\times} S^2$ (S^2 bundle over $\mathbb{C}P^n$) presented in [6][7].

First, we set $\delta^2 = \lambda J^2$, then the coefficient of $1/r^{2n}$ in W vanishes.

Twisting the U(1)×ℝ coordinates as

$$\tilde{\tau} = \tau + J\psi, \tag{7}$$

we obtain

$$g = g_K + (Jd\psi - \sigma)^2, \tag{8}$$

where the metric g_K is a local positive Kähler-Einstein metric in dimension $2n+2$,

$$g_K = \frac{dr^2}{W(r)} + r^2 g_{\mathbb{C}P^n} + r^2 W(r) \left(\frac{d\tilde{\tau}}{J} + A \right)^2, \tag{9}$$

and the Kähler form of g_K is given by $d\sigma / 2\sqrt{\lambda}$,

$$\sigma = \left(1 - \frac{\sqrt{\lambda} r^2}{J} \right) d\tilde{\tau} - \sqrt{\lambda} r^2 A. \tag{10}$$

Thus, as is well known, the metric g in (8) turns out to be locally Sasaki-Einstein. If we write the metric g by the coordinates $(\tau, \tilde{\tau})$, instead of (τ, ψ) or $(\tilde{\tau}, \psi)$, we can eliminate the parameter δ after rescaling $M\delta^2 \to M$ and $J\delta^{-1} \to J$ ‡.

On the other hand, twisting the coordinates as

$$\tilde{\psi} = \psi - \frac{c}{J}\tau, \tag{11}$$

we have

$$g = g_C + \omega(r) \left(d\tau + f(r)(d\tilde{\psi} + A) \right)^2, \tag{12}$$

where

$$g_C = \frac{dr^2}{W(r)} + r^2 g_{\mathbb{C}P^n} + q(r)(d\tilde{\psi} + A)^2, \tag{13}$$

and the components are given by

$$\begin{aligned} \omega(r) &= k^2 r^2 W(r) + (k\sqrt{\lambda} r^2 - 1)^2, \\ f(r) &= \frac{r^2}{\omega(r)} \left(kW(r) + \sqrt{\lambda}(k\sqrt{\lambda} r^2 - 1) \right), \\ q(r) &= \frac{r^2 W(r)}{\omega(r)} \end{aligned} \tag{14}$$

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with $k = (c+1)/J$. The metric g_C is conformally Kähler [7].

The singularities coming from the roots $r = r_i$ of $W = 0$ can be resolved by the restriction of the range of the angle $\tilde{\psi}$; putting $R^2 = 4(r - r_i)/W'(r_i)$ one has in the limit $r \to r_i$,

$$\frac{dr^2}{W(r)^2} + q(r)d\tilde{\psi}^2 \to dR^2 + K_i^2 R^2 d\tilde{\psi}^2, \tag{15}$$

where

$$K_i = \frac{(n+2)\lambda r_i^2 - (n+1)}{k\sqrt{\lambda} r_i^2 - 1}. \tag{16}$$

If we set $\lambda(n+2)/(n+1) = k\sqrt{\lambda}$, that is,

$$c = \frac{n+2}{n+1}\sqrt{\lambda}J - 1, \tag{17}$$

then K_i is independent of r_i. Under a suitable condition on the parameter MJ^2, the corresponding metric g has an SU$(n+1)\times$U$(1)\times$U(1) symmetry, and it reproduces a Sasaki-Einstein metric on a compact manifold given by Gauntlett et al. in [6][7].

We shall comment on the implication of our method in the higher dimensional context. As explained above, the higher dimensional backgrounds are related each other as follows:

$$\begin{array}{ll}
\mathrm{AdS}_p \times S^q & \mathrm{AdS}_p \times M^q_{SE} \\
\quad \updownarrow \text{ Wick rot.} & \quad \uparrow \text{ Wick rot. and } \delta^2 = \lambda J^2 \\
H^p \times \mathrm{dS}_q & H^p \times M^q_{dS} \\
\quad \updownarrow \text{ cosmo.} & \quad \updownarrow \text{ cosmo.} \\
S^p \times \mathrm{AdS}_q & S^p \times M^q_{AdS}
\end{array}$$

where $(p, q) = (5, 5), (4, 7)$, and a p-form flux is associated with them. The left hand side shows that the maximally supersymmetric backgrounds are related to each other. Under the Wick rotation and a sign change of the cosmological constant, the $\mathrm{AdS}_5 \times S^5$ solution in the type-IIB string theory is mapped to itself, while $\mathrm{AdS}_4 \times S^7$ becomes $S^4 \times \mathrm{AdS}_7$. In the right hand side, we have generalized S^q to M^q_{SE}, where M^q_{SE} stands for the q-dimensional Sasaki-Einstein manifold specified by (12). This shows the relation between $\mathrm{AdS}_p \times M^q_{SE}$ and $S^p \times M^q_{AdS}$, where M^q_{AdS} means the q-dimensional Kerr-AdS black hole. It is known that the former solution admits supersymmetry due to the Sasaki-Einstein

structure of M^q_{SE}, and that string/M-theory on $\mathrm{AdS}_p \times M^q_{SE}$ is dual to supersymmetric Yang-Mills theory in $(p-1)$-dimensions. Though the condition $\delta^2 = \lambda J^2$ implies a naked singularity for M^q_{AdS} and cannot be imposed consistently on M^q_{dS}, where M^q_{dS} means the Kerr-dS black hole, it may be interesting to examine string/M-theory on $S^p \times M^q_{AdS}$ and the dual Yang-Mills theory.

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